Central Counterparties and Strategic Reduction of Systemic Risk
Preliminary and Incomplete∗

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Abstract

This paper studies a dynamic network economy where risk averse traders trade multi-laterally over the counter but cannot commit to fulfill their short positions. We show that, although the level of trade is below the first-best, bilateral clearing with collateral can provide an allocation superior to those without collateral. However, with use of collateral, the optimal bilateral clearing contract leads to multiple equilibria, one of which is a scenario of systemic default where defaulting one’s trading partner will trigger the victim to default his trading partners, and so on, causing the spread of contagious default. We show that a simple arrangement with central counterparty clearing (CCP) can eliminate the systemic risk of default contagion, and raise the level of trade, but at the cost of higher level of collateral. We show that whether CCP is essential depends on the opportunity cost of collateral, the discount rate, and the traders’ endowment and their risk aversion.

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1 Introduction

A key element of the regulatory reforms spurred by the Financial Crisis of 2007 and 2008 has been the 2009 G-20 resolution to ensure all standardized Over-the-Counter (OTC) derivative trading is cleared via a central counterparty (CCP) by the end of 2012.\footnote{While this deadline was not met progress has been continuing on ensuring the mandate is eventually met.} This move was due to the dramatic failures of Bear Sterns, Lehman Brothers and the American International Group (AIG). Which demonstrated that a network of exposures in important OTC market and the subsequent counterparty risk can be a significant risk to the financial system in general.

In this paper we look a simple trading market that approximates well the market structure of an OTC inter-dealer market. Specifically, the motivating market for this model is where dealers receive position from clients and then trade amongst themselves to lay-off these positions. The key features of this type of a market are that each dealer trades with multiple members of the market and that by the end of the period each dealer tries to have not net position.

We find that the multilateral nature of this type of market gives rise to multiple equilibrium including an equilibrium where there is a systemic crisis. In this market we find is that CCPs provides a coordination benefit via novation that is different from previous CCP research.

An OTC market with settlement via a CCP is thought to be reduce systemic risk since the CCP interposes itself between buyers and sellers by novating the contract and becoming the buyer for the seller and seller for the buyer. The thinking is that this reduces the systemic risk primarily by simplifying the web of exposures. An additional benefit of allowing netting of exposures so that only net exposures between market participants remain which could
potentially reduce the need for collateral in the financial system. These benefits have been well studied; for example by Duffie and Zhu (2011) in terms of netting effects and by Carapella and Mills (2012) in terms of novation and mutualization of risk. Less studied is the strategic equilibrium effects that CCPs have on trading in the OTC market although one notable exception is Fontaine et al. (2012) which studies how a CCP increases the market power of dealers in the OTC market.

We focus on how a market equilibrium with settlement via a CCP differs from bilateral settlement. Specifically, we develop a benchmark model that is akin to Lyons (1997) in that traders trade amongst themselves multilaterally (thus simultaneously long as well as short the trading of derivatives) but settlement happens bilaterally. A lack of commitment makes collateralized trade superior to uncollateralized trades.

Given the multilateral nature of trade and collateralization we show that the economy is subject to multiple equilibria. The default of one person’s trading partner will cause that person to default on their other counterparty. This is somewhat similar to a multi-lateral bank run. In this model this is the only risk that agents face. The systemic risk exists naturally due to the interplay of limited commitment, risk aversion, and the nature of multi-lateral trades. This risk exists without the need for private information, incomplete market or irrationality of traders in this economy.

When we introduce a CCP into this environment we show that it can eliminate this systemic default risk by removing the risk of this run via novation. In addition, we show that this elimination of systemic risk comes at a collateral cost that is higher than the optimal level. Thus, adopting a CCP or not may involve a trade-off between stability and efficiency in this economy.

In addition to the endogenous response of market results these results provide a strategic reason to have CCP in comparison to the insurance based
reasons previously. This is analogous to the strategic aspect of collateral that Lacker (2001) explores compared to the typical idea of collateral as insurance to exogenous default.

2 The Model

The basic structure in the model is related to Lagos and Wright (2005).

Time is indexed by \( t = 0, 1, \ldots \). In each period there are two sub-periods. A trading period followed by a settlement subperiod.

There are two kinds of goods in this economy, a "derivative" traded in the first period (trading) and "cash" which are both divisible and perishable between periods \( t \).

There are a continuum of traders indexed by \([0, 1]\) who live forever. Each trader discounts the future and has a discount factor \( \beta \in (0, 1) \). An individual trader has the following preferences over derivatives and cash:

\[
\sum_{t=0}^{\infty} \beta^t \{ u(d_t) - l_t + U(x_t) \},
\]

where \( d_t \) and \( l_t \) are the derivatives bought and the derivative underwritten (i.e. sold) in the trading subperiod, \( x_t \) is cash consumption in the settlement subperiod.

We maintain the following standard assumptions on preferences. First, the assumption on derivatives:

**Assumption 1 (Utility for derivatives).** \( u(d) \) is in the set \( \mathbb{C}^2 \) of smooth functions with the first two derivatives which are assumed to be \( u_d > 0, u_{dd} < 0 \). Also, assume \( u_d > 0, u_{dd} < 0, \lim_{d \to 0} u_d(d) = \infty, \) and \( \lim_{d \to \infty} u_d(d) = 0 \).

In other words, we normalize \( u(0) = 0 \) so that traders in the autarky will
have zero payoff in the trading subperiod. We assume the following for cash:

**Assumption 2** (Utility for cash). \( U(x) \) is in the set \( \mathbb{C}^n \) of smooth functions with the following derivatives \( U_x > 0, U_{xx} < 0 \). In addition the function has the following limits \( \lim_{x \to -\infty} x U_x (x) = \infty \), and \( \lim_{x \to x'} U (x) = -\infty \) for all \( x' \leq 0 \).

We also assume the strictly concave payoff cash payoff \( (U(x)) \).

Derivatives are underwritten and traded over-the-counters in the trading subperiod. That is they are traded in a decentralized market where buyers trade bilaterally.

To model the emergence of trading network in this economy, it is assumed that derivative generates payoff for trader \( i \in [0, 1] \) only when it is underwritten by trader \( j = (i + \delta) \mod 1 \), where \( \delta \) is an iid shared by all traders. For all \( i \in [0, 1] \), the trader \( i \) is matched, with certainty, with his derivative buyer \( j = (i - \delta) \mod 1 \) as well as his derivative seller \( j' = (i + \delta) \mod 1 \). Thus traders \( i \) and \( j \) are in the same trading network if and only if there exists \( N \) such that \( j = (i + N\delta) \mod 1 \). We assume that the derivative buyer can make a take-it-or-leave-it offer in the trading subperiod.

Cash in this economy has two roles: to serve as collateral, to help facilitate derivative trade in the trading subperiod; and to serve as a medium of clearing in the settlement subperiod. There is fixed supply of cash in this economy.\(^3\) Each trader is endowed with one unit and \( e \) units of cash in the beginning and the end of the period respectively.

To model the opportunity cost of cash collateral in this economy, consider the key idea is the trader communicates with both buyer and seller simultaneously but that the trader’s buyer and seller do not communicate. This is a plausible assumption since a trading desk at a bank may have multiple traders who each contact a counterparty and can coordinate amongst themselves but the counterparties do not communicate.

\(^2\)For simplicity we have termed the liquid asset cash and assume it has no returns. This is a stand-in for a safe liquid trading instrument such as treasury bills which would be used as collateral or margin for a derivative trade.
period which yields, with certainty, $R > 1$ units of cash in the settlement subperiod for each unit of cash invested. So it is efficient to have all derivative trades be done via credit and have all cash invested in the beginning of the period.

The key friction is that there is no commitment technology to enforce traders to make any future transfer of cash or derivatives. So there is potential tension between efficiency and incentive. Trading derivative on credit is efficient (the gain is $R - 1$) but is subject to the risk of default. To permit the existence of credit, we also assume that there is a coordinating device available such that traders can coordinate among themselves so that exclusion of any trader from future trading is feasible as the harshest punishment.\footnote{One way to motivate this trading relationship would be to assume, as in Chapman et al. (2013), that agents are identifiable within a period but that they are scrambled between period. In addition, an agent who deviates could then cause a “flare” visible to all other agents be detonated causing autarkic.} Then there is maximal credit which is maximum amount a trader prefers to repay rather than to default and be forced into autarky. Any trade beyond where the price is above the maximum possible credit amount implies that an additional spot cash payment, which we interpret as collateral, is needed to settle the difference. This setting could allow a mix of collateral and loan to emerge endogenously as the bilateral clearing arrangement.

3 Bilateral Clearing Contract

We first begin by describing the trading and settlement in an economy where there bilateral trade and settlement. We introduce a central counterparty in the settlement stage in section 5 below. First, we will begin by describing the state variable relevant for a trader in the trading subperiod. We then formulate the trader’s problem given this state variable.

Consider a symmetric situation where all derivative sellers are offered a
contract to sell $d_t$ units of derivative with $\alpha_t \in [0, 1]$ units of cash collateral paid in the trading subperiod and $p_t$ units of cash to be paid later in the settlement subperiod. The offer is denoted as $\{d_t, \alpha_t, p_t\}$. Since every derivative seller is also derivative buyers to some traders, in the symmetric case they hold derivative $d_t$, underwrite derivative $d_t$, receive cash collateral $\alpha_t$ and have $\alpha_t + (1 - \alpha_t) R + e$ cash consumption. Let $V_t \equiv \sum_{s=t}^{\infty} \beta^{s-t} \{ u(d_s) - d_s + U(\alpha_s + (1 - \alpha_s) R + e) \}$ denote the trader’s continuation value in the beginning of period $t$. Then the vector of aggregate states is denoted $S_t \equiv (d_t, \alpha_t, p_t, V_{t+1})$. Traders take the law of motion $S_{t+1} = \Psi(S_t)$ as given, but it will be determined endogenously in equilibrium. Since we are examining stationary equilibrium we suppress the dependence on $t$ unless potential confusion arises.

Given a state $S = (d, \alpha, p, V_{+1})$, consider the trader’s problem where a derivative buyer instead offers $\{d', \alpha', p'\}$ to his derivative seller. In principle, the derivative buyer can default the loan $p'$ in the settlement subperiod with the punishment of remaining in autarky forever, which gives him continuation value $V_A \equiv (1 - \beta)^{-1} U((R + e)$, which is the value of being excluded from the derivative trade forever. On the equilibrium path the rest of traders never default, the incentive constraint that deters the derivative buyer from not paying $p'$ in the settlement subperiod is given by

$$U((1 - \alpha') R + \alpha + p + e - p') + \beta V_{+1} \geq U((1 - \alpha') R + \alpha + p + e) + \beta V_A,$$

(1)

where the trader has $(1 - \alpha') R$ from his cash investment, receives $\alpha$ cash collateral and $p$ cash settlement from his derivative buyer in the trading subperiod and settlement subperiod respectively, and pays his derivative seller $p'$ if he does not default. On the other hand, there is a participation constraint to his derivative seller such that she prefers the offer $\{d', \alpha', p'\}$ rather than rejecting
it:

\[-d' + U((1 - \alpha)R - p + \alpha' + e + p') + \beta V_{+1} \geq \max \left\{ \begin{array}{c} U((1 - \alpha)R - p + e) + \beta V_{+1}, \\ U((1 - \alpha)R + e) + \beta V_A \end{array} \right\}, \]

where his derivative seller has \((1 - \alpha)R + e\) units of cash in the settlement subperiod, pays \(p\) to her derivative buyer, and receives \(\alpha'\) collateral and \(p'\) settlement from the trader in the trading subperiod and in the settlement subperiod respectively. The right hand side is the value of the derivative seller if she rejects the trader’s offer. Off the equilibrium the derivative seller could default the repayment to her derivative seller, which is captured by the max operator.

Given a state \(S = (d, \alpha, p, V_{+1})\), the optimal bilateral clearing contract \(\{d', \alpha', p'\}\) solves:

\[
\max_{d', \alpha' \in [0, 1], p'} \left\{ u(d') - d + U((1 - \alpha')R + \alpha + p + e - p') \right\},
\]

s.t. (1) and (2).

Now we are ready to define the equilibrium in this economy

**Definition 1.** An equilibrium consists of state \(S = (d, \alpha, p, V)\) such that

1. (trader’s maximization) given \(S\), \(\{d, \alpha, p\}\) solves (3)

2. (rational expectation) \(V_{+1} = (1 - \beta)^{-1} [u(d) - d + U(\alpha + (1 - \alpha)R + e)]\)

In the equilibrium, the incentive constraint and the participation constraint become

\[
U((1 - \alpha)R + \alpha + e) \geq U((1 - \alpha)R + \alpha + p + e) + \beta (V_A - V),
\]
\[
U ((1 − \alpha) R + \alpha + e) \geq d + \max \{ U ((1 − \alpha) R - p + e), U ((1 − \alpha) R + e) + \beta (V_A - V) \}.
\] (5)

The following proposition establishes the equilibrium:

**Proposition 1.** Given the maintained assumptions, then there exists an equilibrium.

**Proof.** It is obvious that (2) is always binding. Given \( s = \{\alpha, p\} \), \( d \) and \( V = (1 - \beta)^{-1} [u (d) - d + U (R - (R - 1) \alpha + e)] \), denote \( \Gamma_d (s) : [0, 1] \times \mathbb{R}_+ \rightarrow [0, 1] \times \mathbb{R}_+ \) be the solution of \( \alpha' \) and \( p' \) to the problem (3) given that \( d' \) is given by (2). Since all the constraint (1) are continuous, then \( \Gamma (s) \) is a closed graph. From (1), we have

\[
\beta (V - V_A) \geq U ((1 - \alpha') R + \alpha + p + e) - U ((1 - \alpha') R + \alpha + p + e - p') \\
\geq U_x ((1 - \alpha') R + \alpha + p + e) p' \\
\geq U_x (1 + R + p + e) p'.
\]

Notice that \( V \leq V^* = (1 - \beta)^{-1} [u (d^*) - d^* + U (R + e)] \) for any \( V \), where \( d^* \equiv \arg \max \{ u (d) - d \} \). Then the above inequality implies \( \beta [u (d^*) - d^*] / (1 - \beta) \geq U_x (1 + R + p + e) p' \). Given \( \lim_{x \to \infty} x U_x (x) = \infty \) from the assumption 1, there exists \( p_0 > 0 \) such that \( \beta [u (d^*) - d^*] / (1 - \beta) = U_x (1 + R + p_0 + e) p_0 \).

Thus, for all \( p \in [0, p_0] \), we have

\[
p' \leq \frac{\beta}{1 - \beta} \frac{u (d^*) - d^*}{U_x (1 + R + p + e)} \leq p_0.
\]

Hence, \( \Gamma_d (\alpha, p) \) maps from \([0, 1] \times [0, p_0] \) to \([0, 1] \times [0, p_0] \). Then by the Kakutani fixed point theorem, there exists \( s_d \equiv \{\alpha_d, p_d\} \in [0, 1] \times [0, p_0] \) such that \( s_d = \Gamma_d (s_d) \). Finally, define \( \Delta (d) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) as

\[
\Delta (d) = U ((1 - \alpha_d) R + \alpha_d + e) - \max \{ U ((1 - \alpha_d) R - p_d + e), U ((1 - \alpha_d) R + e) - \beta V_d \},
\]

8
where \( V_d \equiv (1 - \beta)^{-1} [u(d) - d + U((1 - \alpha_d) R + \alpha_d + e)] \). In other words, \( \Delta(d) \) is the value of \( d' \) given by the binding (2) under a given \( d \) and the corresponding fixed point \( s_d \). Then, we have

\[
\Delta(d) \leq U((1 - \alpha_d) R + \alpha_d + e) - U((1 - \alpha_d) R + e) + \beta (V_d - V_A),
\]

\[
\leq U_x((1 - \alpha_d) R + e) \alpha_d + \frac{\beta}{1 - \beta} [u(d) - d],
\]

\[
\leq U_x(e) + \frac{\beta}{1 - \beta} [u(d) - d].
\]

Thus, \( \Delta(d) \) is also bounded. Apply the Kakutani fixed point theorem again, there exists \( d \) such that \( d = \Delta(d) \), hence the equilibrium exists.

The main effort of proving the proposition 1 is to show that there exists a compact set of state \( \{d, \alpha, p\} \) such that the optimal bilateral clearing contract \( \{d', \alpha', p'\} \) is also mapped into that compact set. Then we can apply the Kakutani fixed point theorem. However, the Kakutani fixed point theorem cannot rule out the possibility of multiple equilibria. Nevertheless, it is possible to show that there are some features shared by all equilibria, which is given by the following proposition:

**Proposition 2.** *Given the maintained assumptions. For any equilibrium, we have \( p > 0, d > 0 \) and (2) is binding. Also, there exists \( e_0 \) such that \( \alpha < 1 \) if \( e < e_0 \).*

**Proof.** First of all the Inada condition of \( u \) implies \( d > 0 \). Let \( \lambda_1 \) and \( \lambda_2 \) denote the multiplier to (1) and (2) respectively, \( \lambda_3 \) to \( \alpha \geq 0 \), \( \lambda_4 \) to \( \alpha \leq 1 \) and \( \lambda_5 \) to \( p' \geq 0 \). Then the first order condition of the problem (3) with respect to \( d' \) evaluated in the equilibrium is

\[
\lambda_2 = u_d(d),
\]

which implies \( \lambda_2 \) is always positive, hence (2) to the problem (3) is binding.
The first order condition with respect to \( p' \) and \( \alpha' \) evaluated in the equilibrium are

\[
0 = [u_d(d) - 1 - \lambda_1] U_x(\alpha + (1 - \alpha) R + e) + \lambda_5, \quad (6)
\]

\[
0 = (1 - R) u_d(d) U_x(\alpha + (1 - \alpha) R + e) + \lambda_1 RU_x((1 - \alpha) R + \alpha + p + e) - R\lambda_5 + \lambda_3 - \lambda_4.
\quad (7)
\]

Suppose \( p = 0 \) and \( \alpha > 0 \), then \( \lambda_3 = 0 \). Combining (6) and (7), we have

\[
0 = (1 - R) U_x(\alpha + (1 - \alpha) R + e) - \lambda_4,
\]

which is impossible since the right hand side is negative. Thus we never have \( p = 0 \) and \( \alpha > 0 \). Suppose \( p = 0 \) and \( \alpha = 0 \), then (1) is not binding, hence \( \lambda_1 = 0 \), and (2) implies \( d = 0 \). Then the right hand side of (6) becomes infinity rather than zero. Thus we never have \( p = 0 \) and \( \alpha = 0 \). Combining these cases, in sum we never have \( p = 0 \).

Suppose \( \alpha = 1 \). Since \( p > 0 \), we have \( \lambda_5 = 0 \). From (2), we have

\[
d = U(1 + e) - U(e) - \max \{ U(e - p) - U(e), \beta (V_A - V) \},
\]

\[
= U(1 + e) - U(e) + \min \left\{ \frac{\beta}{1 - \beta} (u(d) - d + U(1 + e) - U(R + e)) \right\},
\]

\[
\geq U(1 + e) - U(e) + \min \left\{ U_x(e) p, \frac{\beta}{1 - \beta} (u(d) - d - U_x(1 + e)(R - 1)) \right\}.
\]

The last term in the last inequality is bounded from below. Then by \( \lim_{d \to \infty} u_d(d) = 0 \) and \( \lim_{x \to 0} U(x) = -\infty \) the of the assumption 1, we have \( d \) greater than \( d^* \) if \( e \) is sufficiently low, since \( -U(e) \) becomes sufficiently large. Then we have \( u_x(d) < u(d^*) \) and the right hand side of (6) is negative rather than zero, which is contradiction. Thus, we prove \( \alpha < 1 \). ■

The proposition characterizes the property of equilibrium. In the equilibrium there is always derivative trade \( (d > 0) \) and credit \( (p > 0) \), but not all of the cash are used for providing collateral \( (\alpha < 1) \) if the cash endowment \( e \) is sufficiently low. The positive derivative trade happens in the equilibrium due to the Inada condition - the marginal payoff of derivative at zero deriv-
tive traded is arbitrarily large. On the other hand, credit is positive in the
equilibrium because zero credit in the equilibrium violates the optimality of
clearing contract. Suppose not. Since derivative must be settled by either
cash collateral or credit, the fact that derivative is trade and zero credit in
the equilibrium means that cash collateral must be posted, ie \( \alpha > 0 \). But
posting costly collateral in exchange for zero credit could never be a part of
the optimal clearing contract.

We are also interested in the equilibrium allocation based on the taxonomy
of whether collateral is posted under the optimal bilateral clearing contract.
Define the first best allocation (with respect to derivative trading only) of this
economy as the amount of derivative trade that maximizes social surplus.

**Definition 2.** The first best allocation is given by \( d = d^* \) where

\[
d^* \equiv \arg\max_d \{ u(d) - d \}.
\]

The above definition of the first best is less restrictive in the sense that it
does not concern the allocation in the settlement subperiod. Otherwise, the
definition would be restricted to the allocation with \( \alpha = 0 \). The following
proposition characterizes the equilibrium when collateral is used

**Proposition 3.** Given the maintained assumptions. For any equilibrium with
\( \alpha > 0 \). Then \( d < (u_d)^{-1}(R) < d^* \) and (1) is binding. If \( \beta R < 1 \), then
\( d > (u_d)^{-1}(\beta^{-1}) \)

**Proof.** Consider an equilibrium with \( \alpha > 0 \) and non-binding (1). Then we
have \( \lambda_1 = \lambda_3 = 0 \). The proposition 1 implies \( \lambda_5 = 0 \). Then the first order
condition (7) becomes

\[
0 = (1 - R) u_d(d) U_x (\alpha + (1 - \alpha) R + e),
\]
which is contradiction since the right hand side is negative. Thus an equilibrium with $\alpha > 0$ implies that (1) is binding and $\lambda_1 \geq 0$. Then $\lambda_5 = 0$ and the first order condition (6) implies $\lambda_1 = u_d(d) - 1 > 0$. Then (7) becomes

$$0 = (1 - R) u_d(d) U_x (\alpha + (1 - \alpha) R + e) + (u_d(d) - 1) RU_x ((1 - \alpha) R + \alpha + p + e) - \lambda_1,$$

$$< (1 - R) u_d(d) U_x (\alpha + (1 - \alpha) R + e) + (u_d(d) - 1) RU_x ((1 - \alpha) R + \alpha + e),$$

$$= [u_d(d) - R] U_x (\alpha + (1 - \alpha) R + e),$$

where the second inequality follows the result $p > 0$ from the proposition 1. Thus, it implies $u_d(d) > R > 1 = u_d(d^*)$. Hence, we have $d < (u_d)^{-1}(R) < d^*$. 

The very existence of collateral in the equilibrium represents an inefficiency due to too little trade in derivatives. An important insight from the proposition 3 is that when traders can default on their credit such that the incentive constraint is binding, then the use of credit is limited and hence derivative is under traded, ie, $d < d^*$. Under these circumstances, posting collateral in the bilateral clearing contract can be welfare improving. A key property of the optimal clearing contract is the use of collateral to relax the incentive constraint in order to raise the level of derivative trade. So collateral is posted only when the incentive constraint is binding and it is desirable to increase the level of derivative trade, ie, when it is below the first best level.

Now we turn to characterizing the equilibrium which also happens to be the first best allocation, if exists. It is given by the following proposition

**Proposition 4.** Given the maintained assumptions. An equilibrium is the first best allocation, if and only if $\alpha = 0$, which is the case if and only if (1) is not binding.

**Proof.** Recall $u_d(d) = 1$ in the first best allocation. Given $p > 0$ and $\lambda_5 = 0$
from the proposition 1, from (6), we have that $u_d(d) = 1$ if and only if $\lambda_1 = 0$, ie, (1) is not binding. Then, altogether with $\lambda_5 = 0$ from the proposition 1, from (7), we have that $u_d(d) = 1$ if and only if

$$\lambda_3 = (R - 1) U_x (\alpha + (1 - \alpha) R + e) + \lambda_4 > 0,$$

ie, $\alpha = 0$. □

Proposition 4 is a stronger version of the welfare theorem in this economy. A version of the first welfare theorem can be interpreted through the proposition 4 as that, any equilibrium with $\alpha = 0$ is also social optimal. On the other hand, a version of the second welfare theorem can be interpreted as follows. Consider a first best allocation. To construct an equilibrium to implement the first best allocation, construct $\alpha = 0$ and $p = p^*$ as the solution to

$$d^* = U (R + e) - U (R + e - p^*).$$

The corresponding incentive constraint is given by

$$\frac{\beta}{1 - \beta} [u (d^*) - d^*] \geq U (R + e + p^*) - U (R + e),$$

where the first-best value is given by $V^* = (1 - \beta)^{-1} [u (d^*) - d^* + U (R + e)]$. Thus, if (9) is not binding, then the proposition 4 states that the first best allocation can be implemented by the equilibrium constructed above.

The next question is then about the existence, which is ignored in the proposition 4: what is a sufficient condition such that there exists an equilibrium which is also the first best allocation? In particular, how do the opportunity cost of collateral $R$ and trader’s wealth $e$ affect the implementability of the first best allocation? The answer is given by the following proposition:
Proposition 5. Given the maintained assumptions. If $U_{xxx} \geq 0$, then there exists $\gamma_0$ such that there exists an equilibrium with the first best allocation if and only if $R + e \geq \gamma_0$.

Proof. Notice that, from proposition 4, there exists an equilibrium with the first best allocation if (9) is satisfied. Total differerentiate (8), we have

$$\frac{dp^*}{d(R + e)} = 1 - \frac{U_x(R + e)}{U_x(R + e - p^*)}.$$ 

Then the differentiation of the right hand side of (9) with respect to $R + e$ is given by

$$U_x(R + p^* + e) \left[ 2 - \frac{U_x(R + e)}{U_x(R + e - p^*)} \right] - U_x(R + e),$$

$$= U_x(R + e + p^*) - U_x(R + e) - \frac{U_x(R + e + p^*)}{U_x(R + e - p^*)} \left[ U_x(R + e) - U_x(R + e - p^*) \right],$$

$$\leq U_x(R + e + p^*) - U_x(R + e) - \frac{U_x(R + e + p^*)}{U_x(R + e - p^*)} \left[ U_x(R + e + p^*) - U_x(R + e) \right],$$

$$= [U_x(R + e + p^*) - U_x(R + e)] \left[ 1 - \frac{U_x(R + e + p^*)}{U_x(R + e - p^*)} \right],$$

$$< 0,$$

where the third line follows the fact that $U_x(R + e + p^*) - U_x(R + e) \leq U_x(R + e) - U_x(R + e - p^*)$ since $U_x$ is convex $(U_{xxx} \geq 0)$ and strictly decreasing. The last line follows from the fact that $p^* > 0$ and $U_x > 0$. In other words, the right hand side of (9) is strictly increasing in $R + e$. Thus, (9) is satisfied if $R + e$ is sufficiently high. 

The condition $U_{xxx} \geq 0$ is not very restrictive: it is assumed in most of the incomplete market literature, and is satisfied under any CARA and any CRRA utility function. A utility function $U$ with $U_{xxx} \geq 0$ features a convex demand curve. To see how the risk aversion affects the equilibrium implementation of the first best allocation, consider a class of CARA utility
function \( U(x) = \frac{\exp(-\gamma x)}{-\gamma} \), where \( \gamma > 0 \) captures the degree of risk aversion - \( U \) is risk-neutral when \( \gamma \to 0 \) and \( U \) is Leontief when \( \gamma \to \infty \). Then combining (8) into the incentive constraint (9), there exists an equilibrium with the first best allocation if

\[
1 \leq \frac{\beta}{\beta + 1} \left[ u(d^*) - d^* \right] \left[ \frac{1}{d^*} + \gamma \exp(\gamma(R + e)) \right],
\]

where the right hand hand is strictly increasing in the risk aversion parameter \( \gamma \). Thus the first best allocation can be implemented in an equilibrium if traders are sufficient averse to risk. When traders are sufficiently more risk averse, they are more sensitive to the downside risk of consumption and less to the upside risk. Hence the marginal benefit of default, which leads to higher consumption immediately in the settlement subperiod, becomes less and the incentive constraint becomes less binding. Thus the first best allocation becomes implementable in the equilibrium.

Proposition 5 states that the first best allocation is implementable in an equilibrium if either the opportunity cost \( R \) of collateral or the cash endowment \( e \) is sufficiently high. It is because of the wealth effect. When either \( R \) or \( e \) is high, on the equilibrium path traders enjoy high level of consumption in the settlement subperiod. Then marginal benefit of default, which leads to higher consumption immediately in the settlement subperiod, could be small and hence defaulting a loan is not desirable even when the level of loan is at its maximal, which is the first best level with zero collateral posted. In this case, the first best allocation can be implemented in an equilibrium.
4 Contagion: Double Deviation v.s. Joint Deviation

One potential risk in this economy is that traders may find profitable to have double deviation: a trader rejects the offer from his derivative buyer in the trading subperiod, trades with his derivative seller and posts the collateral in the trading subperiod but defaults his repayment in the settlement subperiod. Such a double deviation is feasible in this economy rather than in an economy where a trader can only be either a derivative buyer or a derivative seller but not both. In this economy, the single deviation is still ruled out by the optimal bilateral clearing contract: rejection never happens given the participation constraint (2) and default never happens given the incentive constraint (1). However, the double deviation of "reject and default" may be profitable since trader’s value is not concave in both the binary decisions of accept/reject an offer and of default. In particular, "reject and default" is profitable if the following inequality holds

\[ V < u(d') + U((1 - \alpha')R + e) + \beta V_A. \]  

(10)

In the case of joint deviation, the trader still enjoys the payoff of derivative in the trading subperiod, without paying for it nor bearing the cost of underwriting the derivative in trading subperiod. Since the derivative sellers have no bargaining power in this economy, they have zero trade surplus to give up by rejection the offer. Thus, the opportunity cost of the double deviation is only the autarky punishment. Nevertheless, the following proposition rules out the double deviation in the equilibrium and hence can be ignored in the clearing contract:

**Proposition 6.** Given the maintained assumptions. In the equilibrium (10) is
never satisfied.

Proof. In equilibrium, \( d' = d \) and \( \alpha' = \alpha \). Then the right hand side of (10) becomes

\[
\begin{align*}
  u(d) + U((1 - \alpha)R + e) + \beta V_A & \leq u(d) + \max \left\{ U((1 - \alpha)R - p + e) + \beta V, \quad U((1 - \alpha)R + e) + \beta V_A \right\}, \\
& \leq u(d) - d + U((1 - \alpha)R + \alpha + e) + \beta V, \\
& = V,
\end{align*}
\]

where the second inequality follows from the participation constraint (2) and the definition of \( V \). \( \blacksquare \)

However, there could also be a scenario of joint deviation in the settlement subperiod: a trader \( i_1 \) defaults his derivative seller \( i_2 \), which triggers the trader \( i_2 \) to default his derivative seller \( i_3 \), and so on. We refer this scenario as default contagion. In particular, an equilibrium is prone to default contagion if the following inequality is satisfied

\[
U((1 - \alpha)R + \alpha + e - p) + \beta V < U((1 - \alpha)R + e) + \beta V_A. \tag{11}
\]

The left hand side is the value of the solvent trader if his derivative seller defaults the repayment \( p \). The right hand side is the value of the trader if he defaults his derivative seller as well. The following proposition establishes a main result in this paper:

**Proposition 7.** *Given the maintained assumptions. Then (11) is satisfied for any equilibrium with \( \alpha > 0 \).*

*Proof. From the proposition 1, an equilibrium with \( \alpha > 0 \) implies (1) is bind-
\[
U ((1 - \alpha) R + \alpha + e) + \beta V = U ((1 - \alpha) R + \alpha + p + e) + \beta V_A.
\]

Given the strict concavity of \(U\) and \(p > 0\), from the proposition 1, we have

\[
U ((1 - \alpha) R + \alpha + e) > U ((1 - \alpha) R + \alpha + e - p) + U ((1 - \alpha) R + \alpha + p + e) - U ((1 - \alpha) R + \alpha + e).
\]

So substituting \(U ((1 - \alpha) R + \alpha + e)\) of the above inequality into the binding (1) and we establish the proposition.

At the first glance proposition 7 may seem counter-intuitive. An equilibrium is prone to default contagion if collateral is posted. To see the intuition, it is important to notice that in this economy collateral emerges endogenously in order to relax the binding incentive constraint for higher level of derivative trade. Also, every derivative buyer in this economy is also the derivative seller to other traders. Given the risk aversion of cash in the settlement subperiod, the marginal benefit of default his credit is larger when his level of cash in the settlement subperiod is lower - in other words the marginal benefit of default his credit is larger when his derivative buyer defaults than when his derivative buyer does not defaults. In the optimal clearing contract, collateral is posted only until the derivative buyer is indifferent and wont default, given that their derivative buyer does not default on the equilibrium path. So, due to the risk aversion of cash, the derivative buyer will default after his derivative buyer defaults. Eventually the default contagion spreads through the trading network, and it becomes a systemic event.

The following proposition provides a condition for default contagion under the equilibrium with the first best allocation.
Proposition 8. Given the maintained assumptions. Then (11) is satisfied for any equilibrium with the first best allocation only if $\beta < d^*/u(d^*)$. On the other hand, if $\beta > d^*/u(d^*)$, then there exists an equilibrium with the first best allocation such that (11) is not satisfied.

Proof. For any equilibrium with the first best allocation, proposition 4 implies $\alpha = 0$, $V - V_A = [u(d^*) - d^*] / (1 - \beta)$. Thus, substituting (2) into (11) we have (11) is satisfied only if $\beta < d^*/u(d^*)$. On the other hand, suppose $\beta > d^*/u(d^*)$, we have

$$\frac{\beta}{1 - \beta} [u(d^*) - d^*] > d^* = U(R + e) - U(R + e - p^*) = U(R + p^* + e) - U(R + e),$$

where the last inequality follows the result that $p > 0$ from the proposition 2 and $U$ is strictly concave. Thus, (9) is satisfied. By the proposition 4, the first best allocation is also an equilibrium. 

It is interesting to consider two economies, which are different in the opportunity cost of collateral $R$ such that $R + e \geq \gamma_0$ holds in one economy but not the other one. Then the one with higher $R$ is in the equilibrium with $\alpha = 0$, and the one with lower $R$ is in the equilibrium with $\alpha > 0$. If the discount rate $\beta$ is sufficiently high such that $\beta > d^*/u(d^*)$, then, counterintuitively, the economy with higher level of collateral and lower opportunity cost of collateral is prone to the contagion, but not the other one. It is because, the one with higher level of collateral does not mean it is “safer” – on the contrary the very fact that collateral is needed to deter default means that traders have higher incentive to default if their traders partners default. That is why default contagion can happen in the economy with collateral posted and at the same time it does not happen in the economy without collateral.
5 Central Counterparty Clearing

One arrangement to eliminate default contagion is central counterparty (CCP) clearing. Consider a trader who is to pay his derivative seller \( p' \) and receive from his derivative buyer \( p \), the CCP novates these two legs of transaction such that the trader now only needs to pay (receive if negative) \( p' - p \) to the CCP in one settlement, instead of two settlements involving his derivative seller and derivative buyer. In other words, for each trader, the CCP bundles all the transactions and the trader only needs to pay the net settlement. Also, under the CCP, suppose every trader has to post \( \alpha \) cash to the CCP as collateral, which is returned if the trader does not default to the CCP. Thus, at the cost of giving up the collateral and autarky, traders can only default the net payment to the CCP, but not any individual transaction. The incentive constraint of the buyer not to default the CCP (single deviation) becomes

\[
U \left( (1 - \alpha) R + \alpha + p - p' + e \right) + \beta V \geq U \left( (1 - \alpha) R + e \right) + \beta V_A. \tag{12}
\]

Also, traders may find profitable to have the following double deviation: reject any offers from his derivative buyer, trade with his derivative seller and default the CCP. So under CCP we have the following additional incentive constraint to rule out this double deviation:

\[
u (d') - d + U \left( (1 - \alpha) R + \alpha + p - p' + e \right) + \beta V \geq u (d') + U \left( (1 - \alpha) R + e \right) + \beta V_A. \tag{13}
\]
Notice that (12) becomes obsolette when (13) is satisfied. The participation constraint under the CCP becomes

\[-d' + U\left((1 - \alpha)R - p + p' + \alpha + e\right) + \beta V \geq \max\left\{ U\left((1 - \alpha)R + \alpha - p + e\right) + \beta V,\right.\]
\[\left. U\left((1 - \alpha)R + e\right) + \beta V_A \right\}, \tag{14}\]

So, given \(S \equiv \{d_c, \alpha_c, p_c, V_c\}\), the optimal bilateral clearing contract \(\{d', p'\}\) under CCP solves:

\[\max_{d', p'} \{u(d') - d + U\left((1 - \alpha_c)R + \alpha_c + p - p' + e\right)\}, \text{ s.t. } (13) \text{ and } (14), \tag{15}\]

where the continuation value \(V\) is given by

\[V = \frac{u(d_c) - d_c + U\left((1 - \alpha_c)R + \alpha_c + e\right)}{1 - \beta}. \tag{16}\]

Finally, the optimal CCP allocation \(\{d_c, \alpha_c, p_c\}\) solves

\[\max_{d_c, \alpha_c, p_c} V, \text{ s.t. } d' = d_c \text{ and } p' = p_c. \tag{17}\]

The following proposition compares the equilibrium to the optimal CCP allocation:

**Proposition 9.** Given the maintained assumptions. For any any equilibrium without the CCP with \(\alpha \in (0, 1)\), we have \(d < d_c < d^*\). Furthermore, if \(U_{xxx} \geq 0, \frac{U_c((1-\alpha)R+\alpha+e)}{U_x((1-\alpha)R+e)} > 1 - \beta\) and \(d_c \leq (u_d)^{-1}(\beta^{-1})\), then \(\alpha_c > \alpha\).

**Proof.** Let \(\mu_1\) and \(\mu_2\) be the multiplier of (13) and (14) respectively to the problem (15). The first order condition with respect to \(d'\) and \(p'\) evaluated at the optimal CCP allocation is given by \(u_d(d_c) = \mu_2\) and \(1 + \mu_1 = \mu_2\) respectively. Since \(u_d > 0\), we have (14) binding. In the optimal CCP allocation, (13)
is obsolettle when (14) is binding. Then the optimal CCP problem becomes

\[
\max_{d_c, \alpha_c, p_c} \ u(d_c) - d_c + U ((1 - \alpha_c) R + \alpha_c + e) \tag{18}
\]

such that

\[
-d_c + U ((1 - \alpha_c) R + \alpha_c + e)
\]

\[
= \max \left\{ \begin{array}{c}
U ((1 - \alpha_c) R + \alpha_c - p_c + e), \\
U ((1 - \alpha_c) R + e), \\
\frac{-\beta}{1-\beta} \left[ u(d_c) - d_c + U ((1 - \alpha_c) R + \alpha_c + e) - U (R + e) \right] \\
-\beta u(d_c) - d_c + U ((1 - \alpha_c) R + \alpha_c + e) - U (R + e)
\end{array} \right\}. \tag{19}
\]

Since \(p_c\) only appears in (19) but not (18), and \(d_c\) is increasing in \(p_c\) through (19), it is optimal to have \(p_c\) sufficient high such that the two term inside the maximizer on the right hand side of (19) have the same value, ie,

\[
U ((1 - \alpha_c) R + \alpha_c - p_c + e) = U ((1 - \alpha_c) R + e) - \frac{\beta}{1-\beta} [u(d_c) - d_c + U ((1 - \alpha_c) R + \alpha_c + e) - U (R + e)],
\]

hence (19) becomes

\[
\beta u(d_c) - d_c = (1 - \beta) U ((1 - \alpha_c) R + e) - U ((1 - \alpha_c) R + \alpha_c + e) + \beta U (R + e). \tag{20}
\]

Let \(\mu\) denote the multiplier of (20) to (18). Then the first order conditions with respect to \(d_c\) and \(\alpha_c\) are given by

\[
u_d(d_c) - 1 = \mu [1 - \beta u_d(d_c)], \tag{21}
\]

\[
\frac{1 - R^{-1}}{1 - u_d(d_c)^{-1}} = \frac{U(x (1 - \alpha_c) R + e)}{U_z ((1 - \alpha_c) R + \alpha_c + e)}. \tag{22}
\]
Since \( \frac{U((1-\alpha_c)R+e)}{U_x((1-\alpha_c)R+\alpha_c+e)} \geq 1 \), (22) implies \( d_c \geq (u_d)^{-1}(R) \). The right hand side of (22) implies \( \frac{1-R^{-1}}{1-u_d(d_c)} \in (0, \infty) \) and hence \( d_c < (u_d)^{-1}(1) = d^* \). Also, recall \( d < (u_d)^{-1}(R) \) from proposition 3, then we establish \( d < (u_d)^{-1}(R) \leq d_c < d^* \).

To show \( \alpha_c > \alpha \), notice that the participation constraint in the equilibrium with CCP implies

\[
U((1-\alpha)R + \alpha + e) \geq d + U((1-\alpha)R + e) + \beta (V - V_x).
\]

Expanding \( V_x \) and \( V \), the above inequality is equivalent to

\[
\beta u(d) - d - \beta U(R + e) \geq (1 - \beta) U((1-\alpha)R + e) - U((1-\alpha)R + \alpha + e).
\]

Notice that \( \beta u(d) - d \) is strictly concave in \( d \). Then the fact that \( d < d_c \) and the assumption \( d_c \leq (u_d)^{-1}(\beta^{-1}) \) implies \( \beta u(d) - d < \beta u(d_c) - d_c \). Then combining (20), we have

\[
(1 - \beta) U((1-\alpha)R + e) - U((1-\alpha)R + \alpha + e) < (1 - \beta) U((1-\alpha_c)R + e) - U((1-\alpha_c)R + \alpha_c + e).
\]

We need to show the left hand side is increasing in \( \alpha \). Define \( \varphi(y) \equiv (1 - \beta) U((1-y)R + e) - U((1-y)R + y + e) \). Notice that

\[
\varphi'(y) = -R [(1 - \beta) U_x((1-y)R + e) - U_x((1-y)R + y + e)],
\]

\[
\varphi''(y) = R^2 [(1 - \beta) U_{xx}((1-y)R + e) - U_{xx}((1-y)R + y + e)].
\]

The assumption \( U_{xxx} \geq 0 \) implies \( \varphi''(y) < 0 \) for all \( y \in (0,1) \). Also, notice that substituting (21) into the formulation of \( \varphi'(y) \), we have

\[
\varphi'(\alpha_c) = -RU_x((1-\alpha)R + \alpha + e) \left[ -\beta + u_d(d_c)^{-1} - (1 - \beta) R^{-1} \right].
\]

23
Then the assumption $d_c \leq (u_d)^{-1}(\beta^{-1})$ implies $\varphi'(\alpha_c) > 0$. Also, substituting the assumption $\frac{U_x((1-\alpha)R+\alpha+e)}{U_x((1-\alpha)R+e)} > 1 - \beta$ into the formulation of $\varphi'(y)$, we have $\varphi'(\alpha) > 0$. In sum, since $\varphi'(\alpha_c) > 0$, $\varphi'(\alpha) > 0$ and $\varphi''(y) < 0$ for all $y \in (0, 1)$, we have $\varphi'(y) > 0$ for all $y \in [\min(\alpha_c, \alpha), \max(\alpha_c, \alpha)]$. Integrating $\varphi'(y)$ over any interval within $[\min(\alpha_c, \alpha), \max(\alpha_c, \alpha)]$, then we must have the result that for any $y, y' \in [\min(\alpha_c, \alpha), \max(\alpha_c, \alpha)]$, $\varphi(y) > \varphi(y')$ if and only if $y > y'$. Thus the inequality (24) implies $\alpha_c > \alpha$.

Proposition 9 highlights some trade-off involved in settling via a CCP. On one hand, it is clear that a CCP strengthens the stability of the economy by eliminating the systemic risk. It is because CCP settlement can rule out the default contagion, due to novation of the trades. On the other hand, depending on parameters, it is not clear that adopting CCP settlement will raise or reduce the efficiency of the economy.

A CCP helps to reduce the traders’ incentive to default by netting all positions. As a result, the level of derivative trade increases, which was lower when there was no CCP. This was due traders’ incentives to default. This raises the efficiency of the economy. However, novation also invites possible double deviations of traders. They may find it profitable to reject an offer and default on the CCP. Double deviations would not happen if traders were instead settled bilaterally. So to rule out the double deviation, a higher level of collateral has to be posted. But holding collateral is costly in this economy since it forgoes high yield investment. So the efficiency of the economy may decrease after CCP is adopted if the efficiency gain from raising the level of derivative trade is dominated by the efficiency loss from raising the level of costly collateral.
6 Discussion and Conclusion

To come

References


